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PAPER

The method of Φ-Laplace Adomian decomposition for Φ-Caputo fractional Bloch equations

M Shahrezaee[1], M Arabameri[1,*] and A Ahmadian[2,3]

[1] Department of Mathematics, University of Sistan and Baluchestan, Zahedan, Iran
[2] Faculty of Engineering and Natural Sciences, Istanbul Okan University, Istanbul, Turkey
[3] Jadara University Research Center, Jadara University, Jordan
[*] Author to whom any correspondence should be addressed.

E-mail: arabameri@math.usb.ac.ir

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Abstract

This article studies the Bloch equations (BEs), which form a system of macroscopic equations used for the simulation of nuclear magnetization as a function of time, when the relaxation times T_1 and T_2 are given. These equations have been applied to describe nuclear magnetic resonance (NMR), electron spin resonance (ESR), and magnetic resonance imaging (MRI). In this work, we present analytic solutions to the fractional Bloch equations (FBEs). The fractional derivatives in the Bloch equations under consideration are in the sense of Φ-Caputo; we use the Φ-Laplace Adomian decomposition procedure (Φ-LADP) to solve the FBEs. This procedure combines both the Adomian decomposition and Φ-Laplace transform methods. To explain the analytical solutions of the system of Φ-Caputo fractional Bloch equations (Φ-CFBEs) of the order η with known initial conditions, we apply the two-dimensional and three-dimensional phase portraits. We compare these solutions by considering diverse functions in place of $\Phi(t)$ and values of $0 < \eta \leqslant 1$. Finally, to show the usefulness of our proposed method, we discuss the advantages of the new method compared to the existing methods for solving Caputo FBEs.

1. Introduction

The development of integral and differential operators of non-integer orders is closely related to the evolution of integer-order calculus [1]. In recent years, fractional calculus has been extensively studied by many researchers. This is due to the fact that fractional derivatives are essential tools for describing the dynamical treatment of different physical systems. The important feature of these differential operators is that, unlike the integer-order differential operators, they are nonlocal. One of the most remarkable features of fractional differential equations (FDEs) is their ability to describe memory and effectively represent the properties of diverse mathematical models. Also, fractional-order models are more practical and exact compared to integer-order models. In fact, a fractional-order derivative provides a greater degree of independence in these models. Derivatives of arbitrary orders are strong tools for the study of the dynamical treatment of diverse biomaterials and systems.

So far, several fractional operators have been defined. The Riemann–Liouville, Caputo, Hadamard, Erdelyi–Kober, Caputo–Hadamard, and Caputo–Erdelyi–Kober fractional operators are the most remarkable ones to mention [2–5]. Also, the Φ-Caputo derivative introduced by Almeida is the fractional derivative of a function with respect to a function Φ [6]. This fractional derivative is an extra efficient means to model diverse real-world physical phenomena and has the ability to reveal concealed properties. Existence and uniqueness of the solution to a FDE containing the Φ-Caputo derivative were obtained by Almeida *et al* [7].

FDEs provide adequate models in branches such as diffusion processes, damping laws, and other physical phenomena. Compared to integer-order differential equations, FDEs offer numerous advantages in the simulation of main physical processes and dynamic systems [8–10]. Many numerical methods have been developed in recent years to solve FDEs or fuzzy FDEs; examples include the operational matrix technique

[11–13], the differential transform technique [14], the tau method [15], and the variational repetition method [16]. Furthermore, some numerical techniques have been developed for fractional equations containing the Φ-Caputo derivative. These include the method of operational matrix that uses the Φ-shifted Legendre polynomials for Φ-FDEs [17], and the Φ-Haar wavelet technique [18].

Bloch equations are classical, integer-order differential equations that describe the precession of magnetization and its relaxation in relation to space and time. BEs have been extensively used in physics, chemistry, and particularly in NMR, MRI, and ESR. Analysis of time-correlated data acquired from MRI requires a mathematical model to elucidate the correlation among measurable NMR parameters and the progression of disease, such as chemical transformations and diffusion invariants for specific pulse sequences, all of which serve as critical inputs for modeling disease dynamics [19]. Diverse forms of fractional BEs have been considered by many researchers. For instance, Magin *et al* [20] applied the Caputo derivative to fractionate the BEs. Moreover, Magin *et al* [21] proposed the BEs with ηth-order Riemann–Liouville fractional derivative. Also, the authors of the present paper proposed the Bloch equations with ηth-order Φ-Caputo fractional derivative in [22]. Recently, diverse techniques have been developed to approximate the solutions to Bloch equations and fractional Bloch equations [20, 23–26].

Numerous methods exist for solving FDEs, both analytically and numerically. In 1980, Adomian proposed what is now known as the Adomian decomposition procedure. This is an effective technique for acquiring explicit and numerical solutions to systems of differential equations considered in physical problems. The Laplace transform method is a strong technique in applied mathematics and engineering. Among the existing methods, the Laplace–Adomian decomposition procedure has proven to be one of the most effective and descriptive techniques for solving FDEs. This procedure is derived from both the Laplace transform and Adomian decomposition methods. More specifically, this technique can be applied to systems of nonlinear and linear ordinary and partial differential equations of both fractional and classical orders. Recently, some studies used the Laplace decomposition method to solve nonlinear fractional partial differential equations such as the Kaup-Kupershmidt and fractional Caudrey-Dodd-Gibbon-Sawada-Kotera equations [27, 28]. For further details on the Laplace–Adomian decomposition procedure and its applications in solving various models of integer and fractional differential equations, one can refer to the recent studies in [29–31].

In this paper, we concentrate on the coupling of the generalized Laplace transform (GLT) and the Adomian decomposition procedure of FBEs. This results in a procedure which is as strong as Φ-LADP. Φ-LADP produces an analytical solution for Φ-FBEs in the form of a polynomial using only a few repeats. Also, Φ-LADP can solve nonlinear problems without discretization and linearization methods. Using the generalized Laplace transform, we transform FBEs to algebraic equations and decompose nonlinear sentences into sentences of Adomian polynomials. This technique needs less computation work, for this purpose we compare the central processing unit (CPU) times required for obtaining solutions of Φ-CFBEs with the method of Φ-LADP for diverse functions in place of Φ. In summary, we aim to find solutions to systems of Φ-Caputo FBEs by using the Φ-LADP and MATLAB (R2023-a) software. Also, we compare the our results (for the case of $\Phi(t) = t$) with the exact solution and some existing methods for solving Caputo FBEs.

The rest of this paper is organized as follows. The preliminaries of fractional calculus and generalized Laplace transform, the introduction of Bloch equations, and different forms of fractional Bloch equations are provided in section 2. The Φ-LADP of a system of Φ-CFBEs is presented in section 3. Also, in section 4, the obtained results are given graphically. Eventually, the conclusions are presented in section 5.

2. Preliminaries

In this section, we present some essential definitions and mathematical preliminaries of fractional calculus, Φ-CFBEs, and the GLT which are needed to establish our results.

2.1. Fractional calculus

In this subsection, we state the principal definitions and properties of the fractional derivative of a function with respect to another one.

Definition 2.1. [32, 33] Let $\eta > 0$, $\mathcal{D} = [c, d]$ be a finite or infinite interval, $h \in L^1(c, d)$, and $\Phi(t) \in C^1(c, +\infty)$ be a strictly monotone increasing function such that $\Phi(t) \to +\infty$ as $t \to +\infty$, and $\Phi'(t) \neq 0$. The fractional integral of a function h with respect to Φ is defined by

$$\mathcal{I}^{\eta,\Phi}h(t) = \frac{1}{\Gamma(\eta)}\int_c^t \Phi'(\xi)(\Phi(t) - \Phi(\xi))^{\eta-1}h(\xi)d\,\xi, \tag{1}$$

here, $\Gamma(\eta) = \int_0^\infty e^{-u}u^{\eta-1}du$ is the known Gamma function. It can be conveniently verified that the case $\Phi(t) = \ln(t)$ gives $\mathcal{I}^{\eta,\Phi}h(t) = ^H \mathcal{I}^\eta h(t)$ that is the Hadamard fractional integral and the case $\Phi(t) = t$ gives $\mathcal{I}^{\eta,\Phi}h(t) = ^{RL} \mathcal{I}^\eta h(t)$ that is the classical Riemann–Liouville integral [5, 33]:

$$^{H}\mathcal{I}^\eta h(t) = \frac{1}{\Gamma(\eta)}\int_c^t \ln\left(\frac{t}{\xi}\right)^{\eta-1}\frac{h(\xi)}{\xi}d\xi,$$

$$^{RL}\mathcal{I}^\eta h(t) = \frac{1}{\Gamma(\eta)}\int_c^t (t-\xi)^{\eta-1}h(\xi)d\xi.$$

Definition 2.2. [34] Let $AC[c, d]$ be the space of absolutely continuous functions on an interval $[c, d]$. Also, suppose that $\Phi \in C^m[c, d]$ satisfies $\Phi'(t) > 0$ on $[c, d]$. Then,

$$AC_\Phi^m[c, d] = \left\{h\colon [c, d] \to \mathbb{C} \text{ and } h_\Phi^{[m-1]} \in AC[c, d],\quad h_\Phi^{[m-1]} = \left(\frac{1}{\Phi'(t)}\frac{d}{dt}\right)^{m-1} h\right\}.$$

Definition 2.3. [6, 33] Let $\eta > 0$ and $m \in \mathbb{N}$. Also, assume that $\mathcal{D} = (c, d)$ is the interval $-\infty < c < d < \infty$, $\Phi(t) \in C^m(c, d)$ be an increasing and strictly monotone function such that $\Phi(t) \to +\infty$ as $t \to +\infty$, and $\Phi'(t) \neq 0$. The fractional Riemann–Liouville derivative of a function $h \in AC_\Phi^m[c, d]$ with respect to Φ is defined by

$$\mathcal{D}^{\eta,\Phi}h(t) = \left(\frac{1}{\Phi'(t)}\frac{d}{dt}\right)^m \mathcal{I}^{m-\eta,\Phi}h(t).$$

In other words,

$$\mathcal{D}^{\eta,\Phi}h(t) = \frac{1}{\Gamma(m-\eta)}\left(\frac{1}{\Phi'(t)}\frac{d}{dt}\right)^m \int_c^t \Phi'(\xi)(\Phi(t)-\Phi(\xi))^{m-\eta-1}h(\xi)\, d\xi, \tag{2}$$

here $m = [\eta] + 1$ for $\eta \notin \mathbb{N}$, and $\Phi^{(k)} \neq 0$ for $k = 2, \ldots, m$.

It is to be remarked that when $\Phi(t) = \ln t$, (2) becomes the Hadamard fractional derivative. Moreover, for $\Phi(t) = t$, (2) reduces to the classical Riemann–Liouville derivative [1, 4, 5].

Definition 2.4. [6, 33] Let $\eta > 0$ and $m \in \mathbb{N}$. Also, suppose that $\mathcal{D} = (c, d)$ is the interval $-\infty < c < d < \infty$, and $\Phi(t) \in C^m(c, d)$ be an increasing and strictly monotone function such that $\Phi(t) \to +\infty$ as $t \to +\infty$, and $\Phi'(t) \neq 0$. The Caputo fractional derivative of a function $h \in AC_\Phi^m[c, d]$ with respect to Φ is defined by

$$^{C}\mathcal{D}^{\eta,\Phi}h(t) = \mathcal{I}^{m-\eta,\Phi}\left(\frac{1}{\Phi'(t)}\frac{d}{dt}\right)^m h(t), \tag{3}$$

where $m = [\eta] + 1$ for $\eta \notin \mathbb{N}$ and $m = \eta$ for $\eta \in \mathbb{N}$. For a succinct representation, we use the abridged symbol

$$\left(\frac{1}{\Phi'(t)}\frac{d}{dt}\right)^m h(t) = h_\Phi^{[m]}(t),$$

and rewrite (3) as follows:

$$^{C}\mathcal{D}^{\eta,\Phi}h(t) = \frac{1}{\Gamma(m-\eta)}\int_c^t \Phi'(\xi)(\Phi(t)-\Phi(\xi))^{m-\eta-1}h_\Phi^{[m]}(\xi)d\xi.$$

This is also known as the Φ-Caputo fractional derivative.

Definition 2.5. [1, 4, 5] The famous Mittag-Leffler function $E_\eta(p)$, and a usual generalization of $E_\eta(p)$ introduced by Gosta Mittag-Leffler and Wiman, respectively, are defined by

$$E_\eta(p) = \sum_{j=0}^\infty \frac{p^j}{\Gamma(\eta j + 1)},\quad p \in \mathbb{C},\quad \Re(\eta) > 0,$$

$$E_{\eta,\gamma}(p) = \sum_{j=0}^\infty \frac{p^j}{\Gamma(\eta j + \gamma)},\quad p \in \mathbb{C},\quad \Re(\eta),\quad \Re(\gamma) > 0.$$

Here, $\Re(\eta)$ and $\Re(\gamma)$ denote the real parts of η and γ, respectively.

Definition 2.6. [4, 35] Let $\mathbf{A} \in \mathbb{R}^{n\times n}$. The matrix η-exponential function $e_{\eta}^{z\mathbf{A}}$ is defined by

$$e_{\eta}^{z\mathbf{A}} = z^{\eta-1}E_{\eta,\eta}(z^{\eta}\mathbf{A}) = z^{\eta-1}\sum_{j=0}^{\infty}\mathbf{A}^{j}\frac{z^{\eta j}}{\Gamma[(j+1)\eta]},$$

here, $z \in \mathbb{C}\backslash\{0\},\ \eta > 0$.

Lemma 2.7. *[6] Let* $\eta > 0$, $m = [\eta] + 1$ *for* $\eta \notin \mathbb{N}$, $m = \eta$ *for* $\eta \in \mathbb{N}$, $\kappa \in \mathbb{R}$, *and* $\kappa > m$. *Then*

$$\mathcal{I}^{\eta,\Phi}(\Phi(t)-\Phi(c))^{\kappa-1} = \frac{\Gamma(\kappa)}{\Gamma(\kappa+\eta)}(\Phi(t)-\Phi(c))^{\kappa+\eta-1},$$

$$^{C}\mathcal{D}^{\eta,\Phi}(\Phi(t)-\Phi(c))^{\kappa-1} = \frac{\Gamma(\kappa)}{\Gamma(\kappa-\eta)}(\Phi(t)-\Phi(c))^{\kappa-\eta-1}.$$

2.2. The Φ-Laplace transform

In this subsection, we remind the basic definitions and properties of the Laplace transform of h with respect to Φ and the inverse Laplace transform of h with respect to Φ. This generalization of the Laplace transform is due to Jarad and Abdeljawad [34].

Definition 2.8. [32] Suppose that h is a given function defined on $[c, +\infty)(c \in \mathbb{R})$. The Laplace transform of h with respect to Φ is defined by

$$\mathcal{L}_{\Phi}\{h(t)\} = H(s) = \int_{c}^{\infty} e^{-s(\Phi(t)-\Phi(c))}\ h(t)\ \ \Phi'(t)\,dt, \quad s \in \mathbb{C}.$$

In this paper, we also refer to this transform as the Φ-Laplace transform.

Definition 2.9. [32] Suppose that h is a given function defined on $[c, +\infty)(c \in \mathbb{R})$. Then, the inverse Laplace transform of h with respect to Φ is defined by

$$\mathcal{L}_{\Phi}^{-1}\{H(s)\} = h(t) = \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} e^{s(\Phi(t)-\Phi(c))}\ H(s)\,ds, \quad x > 0,\ i^2 = -1.$$

Definition 2.10. [34] An n-dimensional function h: $[0, \infty] \to \mathbb{R}^n$ is said to be of Φ-exponential order $f > 0$ if there exist positive constants M and T such that for all $t > T$,

$$\|h\|_{\infty} = \max_{1\leqslant i\leqslant n}\|h_i\|_{\infty} \leqslant Me^{f\Phi(t)},$$

i.e.

$$h(t) = \mathcal{O}(e^{f\Phi(t)}) \quad as \quad t \to \infty.$$

Theorem 2.11. *[34] If* h: $[0, \infty) \to \mathbb{R}$ *is a piecewise continuous function and is of* Φ*-exponential order* $f > 0$, *where* Φ *is a positive increasing function with* $\Phi(0) = 0$, *then the* Φ*-Laplace transform of* h *exist for* $s > f$.

Theorem 2.12. *[34] Let* $h,\ \Phi$: $[c, \infty) \to \mathbb{R}$ *be real-valued functions, here* $\Phi(t)$ *is continuous,* $\Phi'(t) > 0$ *on* $[0, \infty)$, *and the Laplace transform of* h *with respect to* Φ *exists. Then*

$$\mathcal{L}_{\Phi}\{h(t)\}(s) = \mathcal{L}\{h(\Phi^{-1}(t + \Phi(c)))\}(s),$$

here $\mathcal{L}\{h\}$ *is the classical Laplace transform of* h.

Theorem 2.13. *[34] If the Laplace transform of* h_1: $[c, \infty) \to \mathbb{R}$ *with respect to* Φ *exists for* $s > f_1$ *and the Laplace transform of* h_2: $[c, \infty) \to \mathbb{R}$ *with respect to* Φ *exists for* $s > f_2$, *then for arbitrary constants* α *and* β, *the* Φ*-Laplace transform of* $\alpha h_1 + \beta h_2$ *exists and*

$$\mathcal{L}_{\Phi}\{\alpha h_1(t) + \beta h_2(t)\}(s) = \alpha\mathcal{L}_{\Phi}\{h_1(t)\}(s) + \beta\mathcal{L}_{\Phi}\{h_2(t)\}(s), \quad for\ s > \max\{f_1, f_2\}.$$

Lemma 2.14. *[34] Table 1 presents the* Φ*-Laplace transform of some elementary functions.*

Theorem 2.15. *[34] Let* $\eta > 0$, $\mathcal{D} = [c, d]$ *be a finite or infinite interval and* $h \in AC_{\Phi}^{m}$ *for any* $d > c$. *Then,*

$$\mathcal{L}_{\Phi}\{(\,^{C}\mathcal{D}^{\eta,\Phi}h)(t)\}(s) = s^{\eta}\left[\mathcal{L}_{\Phi}\{h(t)\} - \sum_{l=0}^{m-1}s^{-l-1}(h_{\Phi}^{[l]})(c)\right].$$

2.3. Bloch equations

In this subsection, we introduce classical Bloch equations and some fractional models of Bloch equations. The realization of BEs provides us a basic framework for explaining magnetic resonance phenomena, and dynamic balance and facilitating breakthroughs in various fields such as quantum physics, medical diagnostics, and matters characterization [36, 37]. Classical Bloch equations, introduced by Felix Bloch in his work [38]

Table 1. Generalized Laplace transform of some functions.

$h(t)$	$\mathcal{L}_\Phi\{h(t)\} = H(s)$
1	$\frac{1}{s},\ s > 0$
$(\Phi(t) - \Phi(c))^\eta$	$\frac{\Gamma(\eta+1)}{s^{\eta+1}},\ \Re(\eta) > 0,\ s > 0$
$e^{\lambda\Phi(t)}$	$\frac{1}{s-\lambda},\ s > \lambda$
$E_\eta(\tau(\Phi(t) - \Phi(c))^\eta)$	$\frac{s^{\eta-1}}{s^\eta - \tau},\ \Re(\eta) > 0,\ \lvert\frac{\tau}{s^\eta}\rvert < 1$
$(\Phi(t) - \Phi(c))^{\gamma-1}E_{\eta,\gamma}(\tau(\Phi(t) - \Phi(c))^\eta)$	$\frac{s^{\eta-\gamma}}{s^\eta - \tau},\ \Re(\eta) > 0,\ \lvert\frac{\tau}{s^\eta}\rvert < 1$

published in 1946, can be written as

$$DN_x(t) = \lambda_0 N_y(t) - \frac{N_x(t)}{T_2},$$
$$DN_y(t) = -\lambda_0 N_x(t) - \frac{N_y(t)}{T_2},$$
$$DN_z(t) = \frac{N_0 - N_z(t)}{T_1}, \tag{4}$$

subject to initial conditions $N_x(0) = N_z(0) = 0$ and $N_y(0) = 100$. Here, $N_x(t)$, $N_y(t)$, and $N_z(t)$ denote the system magnetization at the components x, y, and z, respectively, N_0 stands for balance magnetization, λ_0 signifies resonant frequency introduced by the Larmor relation $\lambda_0 = \gamma B_0$, here B_0 represents a static magnetic field in z-component which is constant in time and $\frac{\gamma}{2\pi}$=42.57 $\frac{MHZ}{Tesla}$ is the gyromagnetic ratio for water protons. Also, T_1 and T_2 illustrate the spin-lattice and spin-spin relaxation times, respectively.

So far, some fractional variants have been suggested for the BEs. Magin *et al* [20] introduced the FBEs

$$^{C}\mathcal{D}^\eta N_x(t) = \lambda_0^* N_y(t) - \frac{N_x(t)}{T_2'}, \tag{5}$$

$$^{C}\mathcal{D}^\eta N_y(t) = -\lambda_0^* N_x(t) - \frac{N_y(t)}{T_2'}, \tag{6}$$

$$^{C}\mathcal{D}^\eta N_z(t) = \frac{N_0 - N_z(t)}{T_1'},\quad 0 < \eta \leqslant 1, \tag{7}$$

with initial conditions $N_x(0) = N_z(0) = 0$, and $N_y(0) = 100$, where $^{C}\mathcal{D}^\eta$ is the Caputo fractional derivative of order η. Also, $\lambda_0^* = \frac{\lambda_0}{\sigma_2^{\eta-1}}$, $\frac{1}{T_1'} = \frac{\sigma_1^{1-\eta}}{T_1}$ and $\frac{1}{T_2'} = \frac{\sigma_2^{1-\eta}}{T_2}$ have the same unit $(\text{sec})^{-\eta}$, and σ_1, σ_2 are fractional time constants to retain a consistent set of units for the magnetization. This pattern is used to study the spin dynamics and magnetization relaxation in the ordinary instance of a singular spin particle at resonance within a stationary electromagnetic field. Velasco *et al* [39] introduced another variant of fractional Bloch equations. Moreover, several fractional generalizations of BEs incorporating a delay have been proposed [20, 40–44].

Theorem 2.16. *The Caputo FBEs (equations (5), (6), and (7)) can be rewritten as*

$$^{C}\mathcal{D}^\eta G(t) = \mathrm{A}G(t) + Q(t), \tag{8}$$

$$G(0) = \bar{N}(0), \tag{9}$$

where

$$G(t) = \begin{bmatrix} N_x(t) \\ N_y(t) \\ N_z(t) \end{bmatrix},\qquad \mathrm{A} = \begin{bmatrix} -\frac{1}{T_2'} & \lambda_0^* & 0 \\ -\lambda_0^* & -\frac{1}{T_2'} & 0 \\ 0 & 0 & -\frac{1}{T_1'} \end{bmatrix} \in \mathbb{R}^{3\times 3},\qquad Q(t) = \begin{bmatrix} 0 \\ 0 \\ \frac{N_0}{T_1'} \end{bmatrix},\qquad \bar{N}(0) = \begin{bmatrix} 0 \\ 100 \\ 0 \end{bmatrix}.$$

equations (8), (9) have a unique solution as follows:

$$G(t) = \int_0^t e_\eta^{(t-\tau)\mathrm{A}}[Q(\tau) + \mathrm{A}\bar{N}(0)]d\tau + \bar{N}(0)$$
$$= \int_0^t e_\eta^{(t-\tau)\mathrm{A}}Q(\tau)d\tau + [\mathrm{A}t^\eta E_{\eta,\eta+1}(t^\eta \mathrm{A}) + I]\bar{N}(0).$$

Proof. See [4, 35]. □

A generalization of FBEs (Φ-CFBEs) has been presented and solved in [22] as follows:

$$ {}^{C}\mathcal{D}^{\eta,\Phi}N_x(t) = \lambda_0^* N_y(t) - \frac{N_x(t)}{T_2'}, \tag{10}$$

$$ {}^{C}\mathcal{D}^{\eta,\Phi}N_y(t) = -\lambda_0^* N_x(t) - \frac{N_y(t)}{T_2'}, \tag{11}$$

$$ {}^{C}\mathcal{D}^{\eta,\Phi}N_z(t) = \frac{N_0 - N_z(t)}{T_1'}, \quad 0 < \eta \leqslant 1. \tag{12}$$

Where, the initial conditions are $N_x(0) = N_z(0) = 0$, and $N_y(0) = 100$. Also, $\lambda_0^* = \frac{\lambda_0}{\rho_2^{\eta-1}}, \frac{1}{T_1'} = \frac{\rho_1^{1-\eta}}{T_1}$ and $\frac{1}{T_2'} = \frac{\rho_2^{1-\eta}}{T_2}$ have the same unit $(\mathrm{sec})^{-\eta}$. The fractional time constants ρ_1, ρ_2 are used to retain a consistent set of units for the magnetization.

3. Φ-LADP for the system of Φ-CFBEs

The aim of this section is to discover the analytic solutions for Φ-CFBEs using the Φ-Laplace Adomian decomposition procedure. Certainly, both the Adomian decomposition and generalized Laplace transform methods are powerful tools for finding solutions of FDEs.

Theorem 3.1. *The Φ-Caputo FBEs (equations (10), (11) and(12)) can be rewritten as*

$$ {}^{C}\mathcal{D}^{\eta,\Phi}G(t) = AG(t) + Q(t), \tag{13}$$

where

$$ G(t) = \begin{bmatrix} N_x(t) \\ N_y(t) \\ N_z(t) \end{bmatrix}, \qquad \boldsymbol{A} = \left[\begin{array}{cc|c} -\frac{1}{T_2'} & \lambda_0^* & 0 \\ -\lambda_0^* & -\frac{1}{T_2'} & 0 \\ \hline 0 & 0 & -\frac{1}{T_1'} \end{array}\right], \qquad Q(t) = \begin{bmatrix} 0 \\ 0 \\ \frac{N_0}{T_1'} \end{bmatrix}, $$

and

$$ \mathbf{B} = \begin{bmatrix} -\frac{1}{T_2'} & \lambda_0^* \\ -\lambda_0^* & -\frac{1}{T_2'} \end{bmatrix}, $$

with initial conditions

$$ N_x(0) = N_z(0) = 0, $$
$$ N_y(0) = 100. $$

Here, $0 < \eta \leqslant 1$*. Also,* $\lambda_0^* = \frac{\lambda_0}{\rho_2^{\eta-1}}, \frac{1}{T_1'} = \frac{\rho_1^{1-\eta}}{T_1}$ *and* $\frac{1}{T_2'} = \frac{\rho_2^{1-\eta}}{T_2}$ *have the solutions*

$$ R(t) = \begin{bmatrix} N_x(t) \\ N_y(t) \end{bmatrix} = E_{\eta,1}[\mathbf{B}(\Phi(t) - \Phi(c))^{\eta}].\, \mathbf{M}, $$

$$ N_z(t) = \sum_{m=1}^{\infty}\sum_{k=1}^{m} (-1)^{k+1}\frac{N_0}{(T_1')^k}.\frac{(\Phi(t) - \Phi(c))^{k\eta}}{\Gamma(k\eta + 1)}, \qquad \forall\ m \geqslant 1, $$

where $\mathbf{M} = \begin{bmatrix} 0 \\ 100 \end{bmatrix}$ *is the vector of initial conditions,* $N_x(0) = 0$ *and* $N_y(0) = 100$.

Proof. To gain the solutions of $N_x(t)$ and $N_y(t)$, we apply the GLT, using theorem 2.15, to equations (10) and (11) as follows.

$$ \begin{cases} \mathcal{L}_\Phi\{N_x(t)\} = \frac{1}{s^\eta}.\, \mathcal{L}_\Phi\{\lambda_0^* N_y(t) - \frac{1}{T_2'}N_x(t)\}, \\ \mathcal{L}_\Phi\{N_y(t)\} = \frac{1}{s^\eta}.\, \mathcal{L}_\Phi\{-\lambda_0^* N_x(t) - \frac{1}{T_2'}N_y(t)\} + \frac{100}{s}. \end{cases} \tag{14}$$

Now, by using the inverse generalized Laplace transform we obtain

$$ \begin{cases} N_x(t) = \mathcal{L}_\Phi^{-1}\Big(\frac{1}{s^\eta}.\, \mathcal{L}_\Phi\{\lambda_0^* N_y(t) - \frac{1}{T_2'}N_x(t)\}\Big), \\ N_y(t) = \mathcal{L}_\Phi^{-1}\Big(\frac{1}{s^\eta}.\, \mathcal{L}_\Phi\{-\lambda_0^* N_x(t) - \frac{1}{T_2'}N_y(t)\}\Big) + 100. \end{cases} \tag{15}$$

According to the Adomian decomposition procedure, we assume

$$N_x(t) = \sum_{m=0}^{\infty} N_{x,m}(t),$$
$$N_y(t) = \sum_{m=0}^{\infty} N_{y,m}(t).$$

Then, using series, the equations (15) can be rewritten in the form

$$\begin{cases} \sum_{m=0}^{\infty} N_{x,m}(t) = \mathcal{L}_{\Phi}^{-1}\left(\frac{1}{s^{\eta}} \cdot \mathcal{L}_{\Phi}\{\lambda_0^* \sum_{m=0}^{\infty} N_{y,m}(t) - \frac{1}{T_2'} \sum_{m=0}^{\infty} N_{x,m}(t)\}\right), \\ \sum_{m=0}^{\infty} N_{y,m}(t) = \mathcal{L}_{\Phi}^{-1}\left(\frac{1}{s^{\eta}} \cdot \mathcal{L}_{\Phi}\{-\lambda_0^* \sum_{m=0}^{\infty} N_{x,m}(t) - \frac{1}{T_2'} \sum_{m=0}^{\infty} N_{y,m}(t)\}\right) + 100. \end{cases} \tag{16}$$

Therefore, the system of equations (16) leads to the recursive algorithm defined by

$$\begin{cases} N_{x,0}(t) = 0, \\ N_{x,m}(t) = \mathcal{L}_{\Phi}^{-1}\left(\frac{1}{s^{\eta}} \cdot \mathcal{L}_{\Phi}\{\lambda_0^* N_{y,m-1}(t) - \frac{1}{T_2'} N_{x,m-1}(t)\}\right), \qquad \forall\ m \geqslant 1, \end{cases} \tag{17}$$

and

$$\begin{cases} N_{y,0}(t) = 100, \\ N_{y,m}(t) = \mathcal{L}_{\Phi}^{-1}\left(\frac{1}{s^{\eta}} \cdot \mathcal{L}_{\Phi}\{-\lambda_0^* N_{x,m-1}(t) - \frac{1}{T_2'} N_{y,m-1}(t)\}\right), \qquad \forall\ m \geqslant 1. \end{cases} \tag{18}$$

We rewrite the recursive relations (17) and (18) using theorem 2.13 and lemma 2.14 as follows.

$$\begin{cases} N_{x,0}(t) = 0, \\ N_{x,1}(t) = \mathcal{L}_{\Phi}^{-1}\left(\frac{1}{s^{\eta}} \cdot \mathcal{L}_{\Phi}\{100\ \lambda_0^*\}\right) = 100\ \lambda_0^* \cdot \frac{[\Phi(t) - \Phi(c)]^{\eta}}{\Gamma(\eta+1)}, \\ N_{x,2}(t) = \mathcal{L}_{\Phi}^{-1}\left(\frac{1}{s^{\eta}} \cdot \mathcal{L}_{\Phi}\left[\frac{-200\ \lambda_0^*}{T_2'} . \frac{(\Phi(t) - \Phi(c))^{\eta}}{\Gamma(\eta+1)}\right]\right) = \frac{-200\ \lambda_0^*}{T_2'} \cdot \frac{[\Phi(t) - \Phi(c)]^{2\eta}}{\Gamma(2\eta+1)}, \\ \quad \vdots \end{cases}$$

and

$$\begin{cases} N_{y,0}(t) = 100, \\ N_{y,1}(t) = \mathcal{L}_{\Phi}^{-1}\left(\frac{1}{s^{\eta}} \cdot \mathcal{L}_{\Phi}\{\frac{-100}{T_2'}\}\right) = \frac{-100}{T_2'} \cdot \frac{[\Phi(t) - \Phi(c)]^{\eta}}{\Gamma(\eta+1)}, \\ N_{y,2}(t) = \mathcal{L}_{\Phi}^{-1}\left(\frac{1}{s^{\eta}} \cdot \mathcal{L}_{\Phi}\left[\left(-100\ (\lambda_0^*)^2 . \frac{(\Phi(t) - \Phi(c))^{\eta}}{\Gamma(\eta+1)}\right) + \left(\frac{100}{(T_2')^2} \cdot \frac{(\Phi(t) - \Phi(c))^{\eta}}{\Gamma(\eta+1)}\right)\right]\right) \\ \quad = \left(-100\ (\lambda_0^*)^2 + \frac{100}{(T_2')^2}\right) \cdot \frac{[\Phi(t) - \Phi(c)]^{2\eta}}{\Gamma(2\eta+1)}, \\ \quad \vdots \end{cases}$$

Eventually, when the described algorithm is enough stated, one obtains the closed-form solution

$$R(t) = \begin{bmatrix} N_x(t) \\ N_y(t) \end{bmatrix} = E_{\eta,1}[\mathbf{B}(\Phi(t) - \Phi(c))^{\eta}].\ \mathbf{M}.$$

To obtain $N_z(t)$, we use the same method. First, we apply the GLT to the both sides of equation (12) to obtain

$$\mathcal{L}_{\Phi}\{N_z(t)\} = \frac{1}{s^{\eta}} \cdot \mathcal{L}_{\Phi}\left(\frac{N_0 - N_z(t)}{T_1'}\right).$$

By applying the inverse GLT, we obtain

$$N_z(t) = \mathcal{L}_{\Phi}^{-1}\left[\frac{1}{s^{\eta}} \cdot \mathcal{L}_{\Phi}\left(\frac{N_0 - N_z(t)}{T_1'}\right)\right],$$

and assuming

$$N_z(t) = \sum_{m=0}^{\infty} N_{z,m}(t), \tag{19}$$

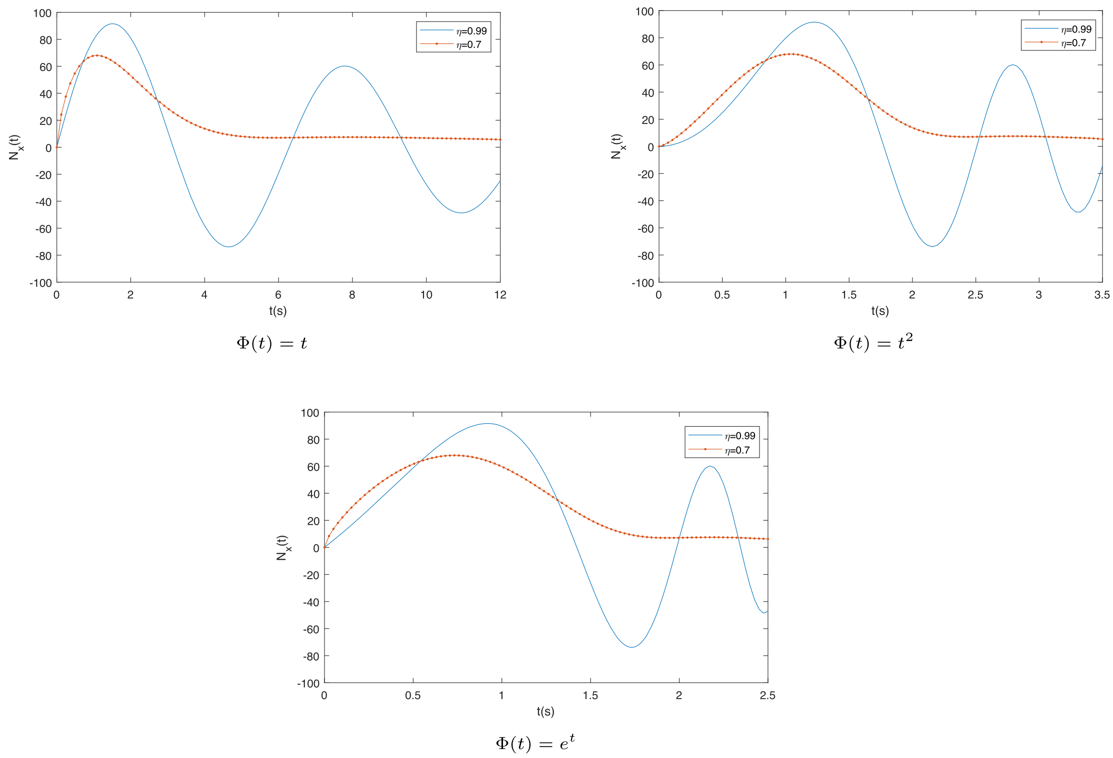

Figure 1. Plots of the $N_x(t)$ solution of the system of equations (13) using Φ-LADP for different functions of $\Phi(t)$ and $\eta = 0.99,\ 0.7$, $T_2' = 20\ (ms)^\eta$, $N_x(0) = 0$, $\lambda_0^* = 1$.

we obtain

$$\sum_{m=0}^{\infty} N_{z,m}(t) = \mathcal{L}_{\Phi}^{-1}\left[\frac{1}{s^\eta} . \mathcal{L}_{\Phi}\left(\frac{N_0 - \sum_{m=0}^{\infty} N_{z,m}(t)}{T_1'}\right)\right]. \tag{20}$$

Then, the relation (20) leads to the recursive relation

$$\begin{cases} N_{z,0}(t) = 0, \\ N_{z,m}(t) = \mathcal{L}_{\Phi}^{-1}\left[\frac{1}{s^\eta} . \mathcal{L}_{\Phi}\left(\frac{N_0 - N_{z,m-1}(t)}{T_1'}\right)\right], \quad \forall\ m \geqslant 1. \end{cases} \tag{21}$$

By rewriting the system (21), we obtain

$$\begin{cases} N_{z,0}(t) = 0, \\ N_{z,1}(t) = \mathcal{L}_{\Phi}^{-1}\left[\frac{1}{s^\eta} . \mathcal{L}_{\Phi}\left(\frac{N_0}{T_1'}\right)\right] = \frac{N_0}{T_1'} . \frac{[\Phi(t) - \Phi(c)]^\eta}{\Gamma(\eta+1)}, \\ N_{z,2}(t) = \mathcal{L}_{\Phi}^{-1}\left[\frac{1}{s^\eta} . \mathcal{L}_{\Phi}\left(\frac{N_0}{T_1'} - \left(\frac{N_0}{(T_1')^2} . \frac{[\Phi(t) - \Phi(c)]^\eta}{\Gamma(\eta+1)}\right)\right)\right] \\ \qquad = \left(\frac{N_0}{T_1'} . \frac{[\Phi(t) - \Phi(c)]^\eta}{\Gamma(\eta+1)}\right) - \left(\frac{N_0}{(T_1')^2} . \frac{[\Phi(t) - \Phi(c)]^{2\eta}}{\Gamma(2\eta+1)}\right), \\ N_{z,3}(t) = \mathcal{L}_{\Phi}^{-1}\left[\frac{1}{s^\eta} . \mathcal{L}_{\Phi}\left(\frac{N_0}{T_1'} - \left(\frac{N_0}{(T_1')^2} . \frac{[\Phi(t) - \Phi(c)]^\eta}{\Gamma(\eta+1)}\right) + \left(\frac{N_0}{(T_1')^3} . \frac{[\Phi(t) - \Phi(c)]^{2\eta}}{\Gamma(2\eta+1)}\right)\right)\right] \\ \qquad = \left(\frac{N_0}{T_1'} . \frac{[\Phi(t) - \Phi(c)]^\eta}{\Gamma(\eta+1)}\right) - \left(\frac{N_0}{(T_1')^2} . \frac{[\Phi(t) - \Phi(c)]^{2\eta}}{\Gamma(2\eta+1)}\right) + \left(\frac{N_0}{(T_1')^3} . \frac{[\Phi(t) - \Phi(c)]^{3\eta}}{\Gamma(3\eta+1)}\right), \\ \qquad \vdots \end{cases} \tag{22}$$

We summarize equations (22) as

$$\begin{cases} N_{z,0}(t) = 0, \\ N_{z,m}(t) = \sum_{k=1}^{m} (-1)^{k+1} \frac{N_0}{(T_1')^k} . \frac{(\Phi(t) - \Phi(c))^{k\eta}}{\Gamma(k\eta+1)}, \quad \forall\ m \geqslant 1. \end{cases} \tag{23}$$

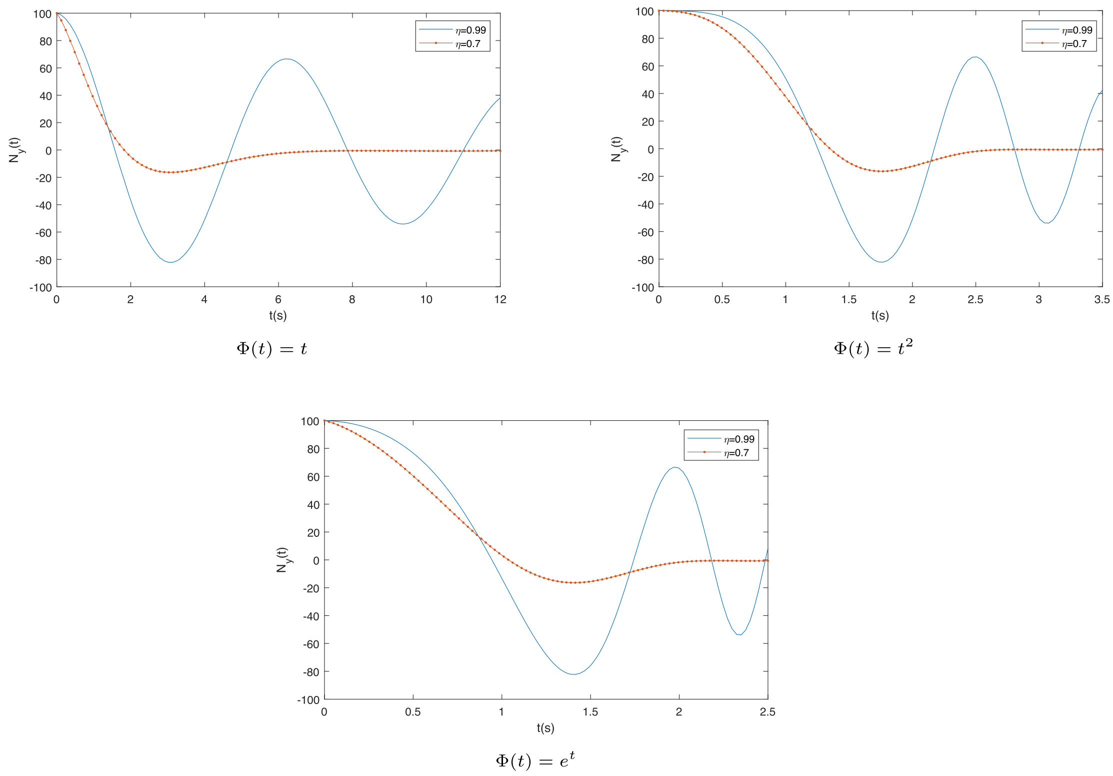

Figure 2. Plots of the $N_y(t)$ solution of the system of equations (13) using Φ-LADP for different functions of $\Phi(t)$ and $\eta = 0.99,\ 0.7$, $T_2' = 20\ (ms)^{\eta}, N_y(0) = 100,\ \lambda_0^* = 1$.

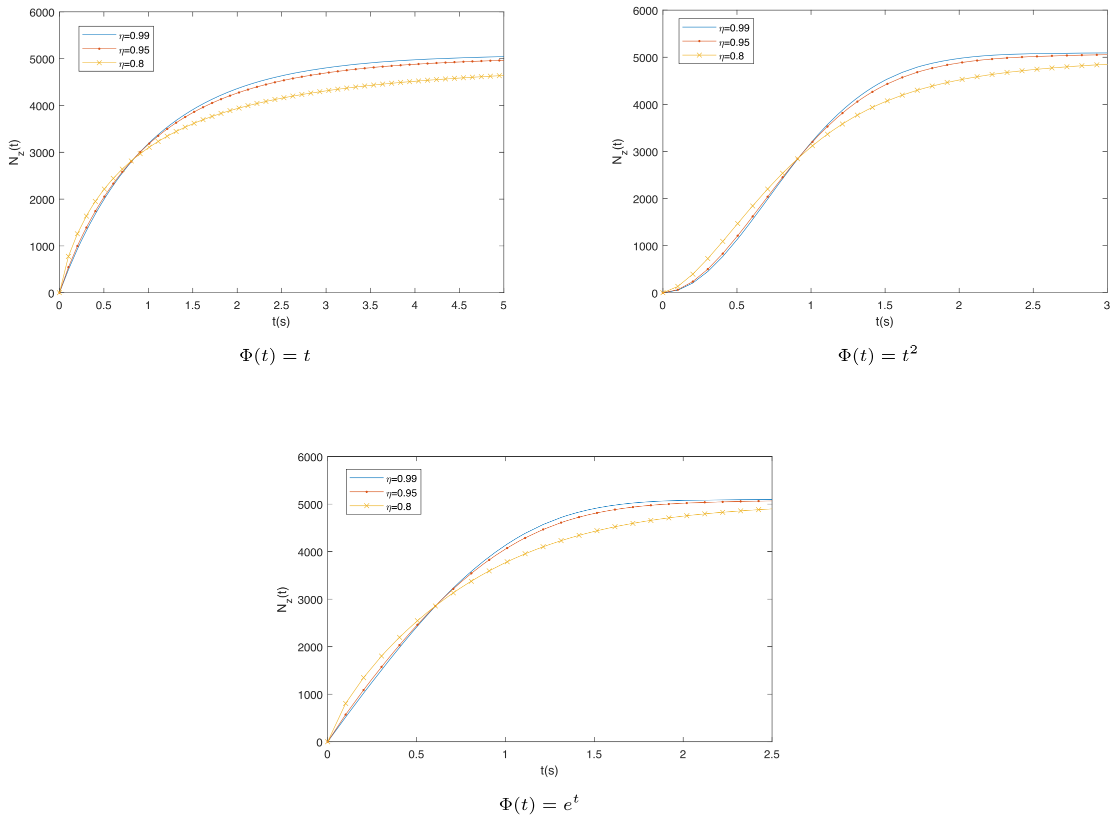

Figure 3. Plots of the $N_z(t)$ solution of the system of equations (13) using Φ-LADP for different functions of $\Phi(t)$ and $\eta = 0.99,\ 0.95,\ 0.8,\ T_1' = 1\ (s)^{\eta}, N_y(0) = 0, N_0 = 100,\ \lambda_0^* = 1$.

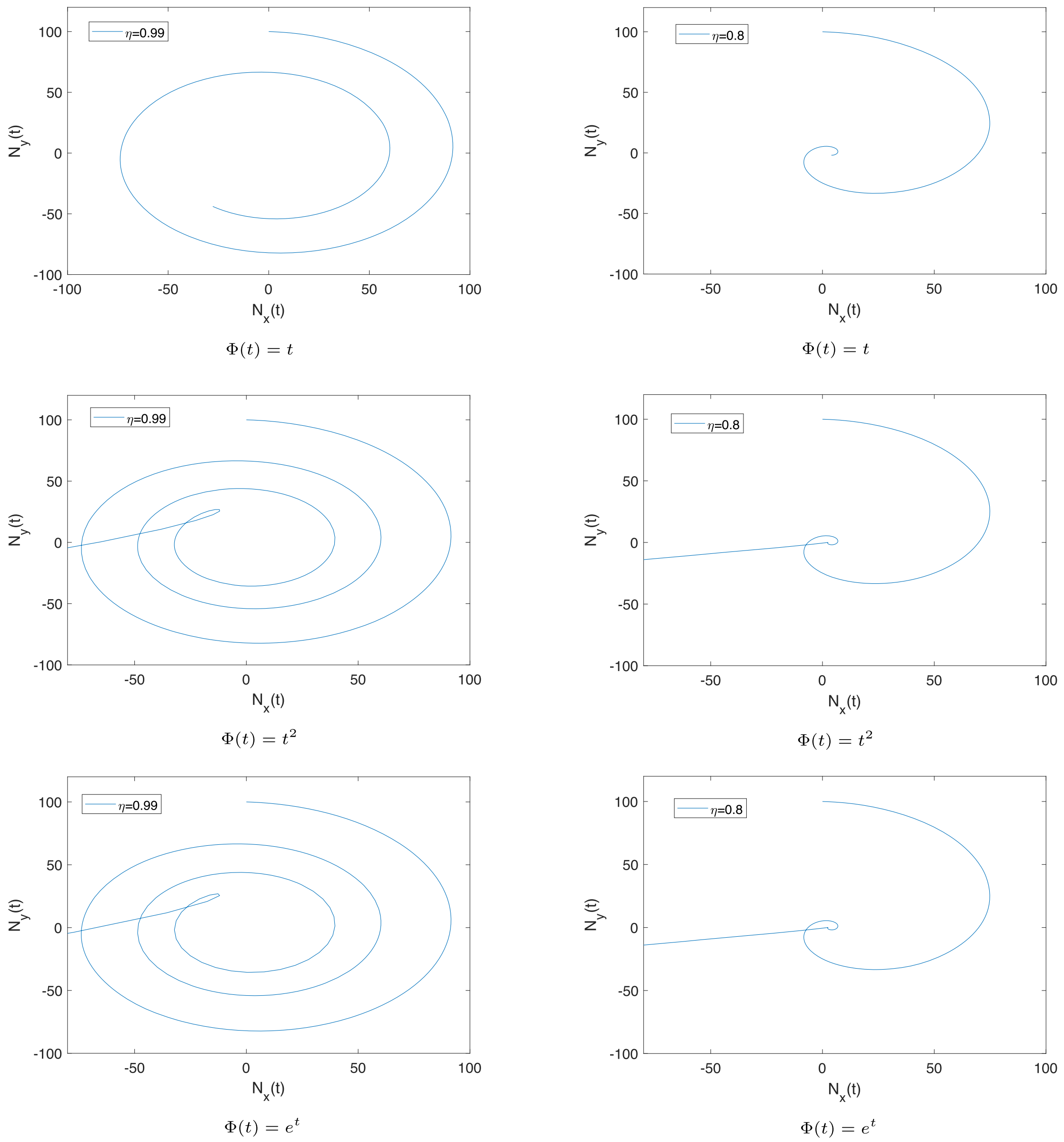

Figure 4. The two-dimensional phase portraits solutions of system (13), $N_x(t)$, and $N_y(t)$, for different values of η and different functions of $\Phi(t)$ for $T_2' = 20\,(ms)^{\eta}$, $N_y(0) = 100$, $N_x(0) = 0$, $\lambda_0^* = 1$. The spiral trajectories illustrate the decay of the transvers magnetization components in the complex plan. The decay in transverse magnetization increases by reducing the fractional derivative order.

By using the relations (19) and (23), we get the solution $N_z(t)$:

$$N_z(t) = \sum_{m=1}^{\infty}\sum_{k=1}^{m}(-1)^{k+1}\frac{N_0}{(T_1')^k}.\frac{(\Phi(t) - \Phi(c))^{k\eta}}{\Gamma(k\eta + 1)}, \qquad \forall\ m \geqslant 1.$$

□

4. Results and discussion

In this section, the analytical results are investigated for the system of Φ-CFBEs. We graphically explain these results for diverse functions in place of $\Phi(t)$ and different values of η using MATLAB, and represent the dynamical behavior of the system by two-dimensional phase portraits and three-dimensional phase portraits. Then, we compare the exact and approximate solutions of equations (13) (for the case of $\Phi(t) = t$) obtained by our method and the proposed methods in [26], [43], and [25]. Also, we compare the maximum absolute error (MAE) the mentioned methods for $\Phi(t) = t$, and $\eta = 1$. Finally we compare the elapsed times to obtain the solution of system (13) by our method for diverse functions in place of Φ.

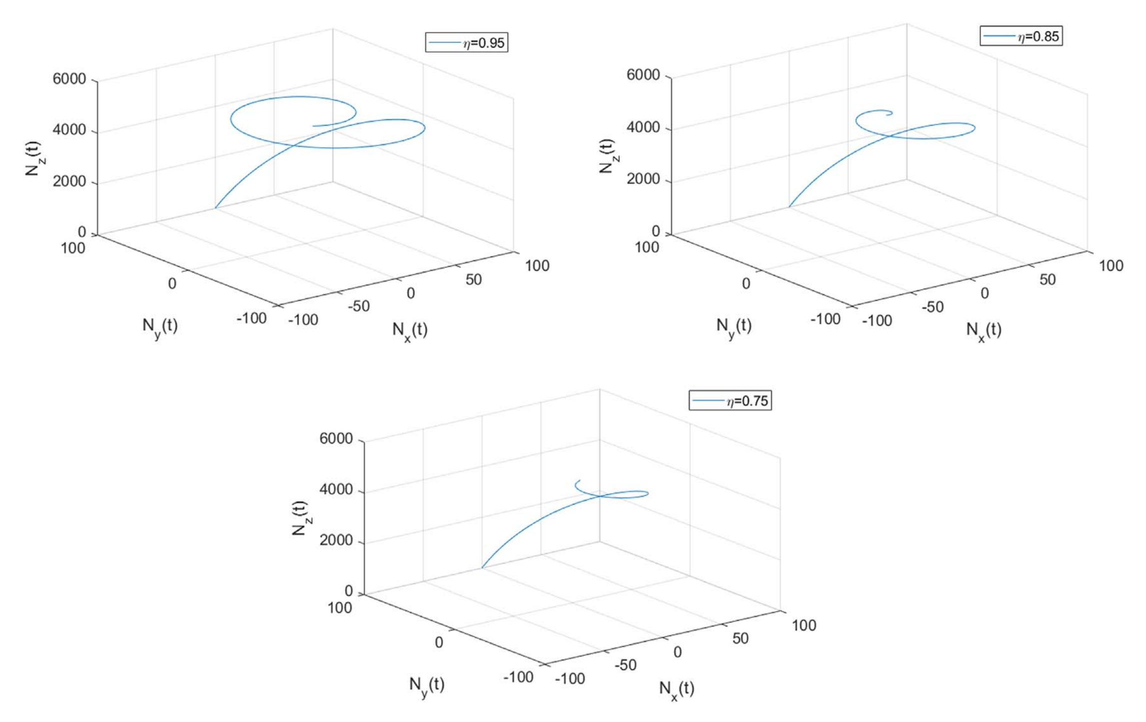

Figure 5. The three-dimensional phase portraits of the solution of system (13), for different values of η and $\Phi(t) = t$ with initial conditions $N_x(0) = N_z(0) = 0,\ N_y(0) = 100$ and $T_1' = 1\ (s)^\eta$, $T_2' = 20\ (ms)^\eta$, $\lambda_0^* = 1$ converging to the balance value N_0. It is evident that reducing the value of η serves as a stabilizing mechanism for the system described by equations (13).

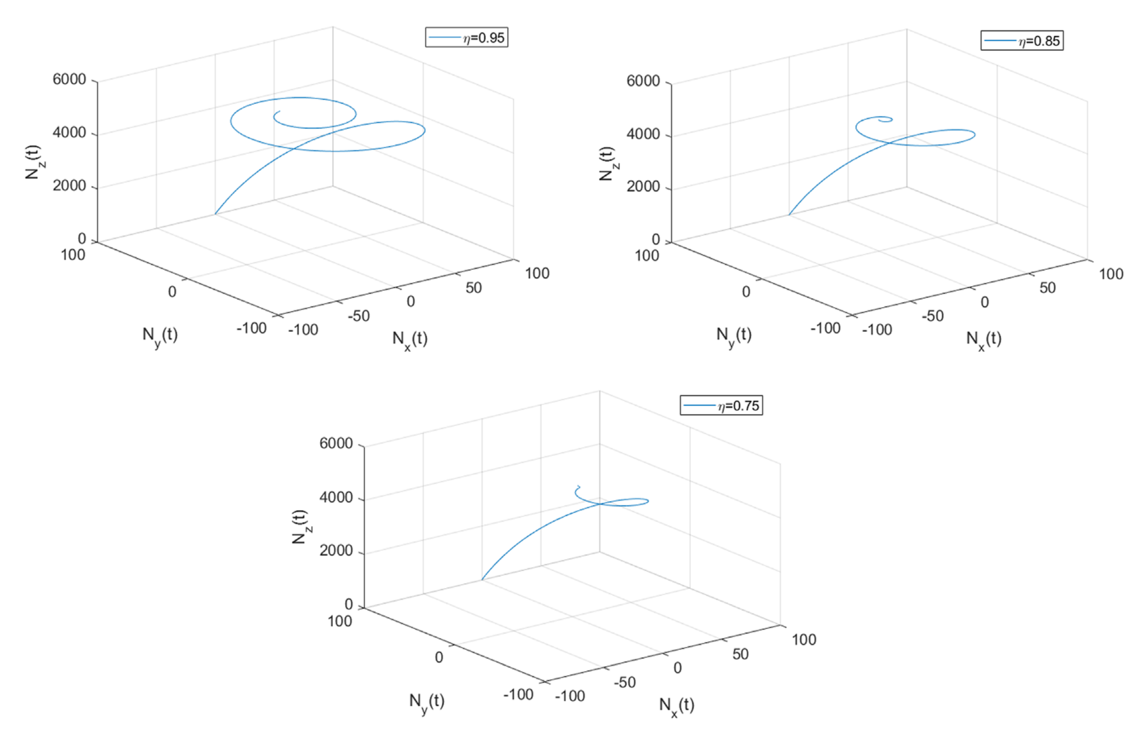

Figure 6. The three-dimensional phase portraits of the solution of system (13), for different values of η and $\Phi(t) = t^2$ with initial conditions $N_x(0) = N_z(0) = 0,\ N_y(0) = 100$ and $T_1' = 1\ (s)^\eta$, $T_2' = 20\ (ms)^\eta$, $\lambda_0^* = 1$ converging to the balance value N_0. It is evident that reducing the value of η serves as a stabilizing mechanism for the system described by equations (13).

In the structure of all the figures, the following parameters have been applied to solve the system of equations (13).

$$T_1' = 1\ (s)^\eta,\ \ T_2' = 20\ (ms)^\eta,\ \ N_0 = 100,\ \ \lambda_0^* = 1.$$

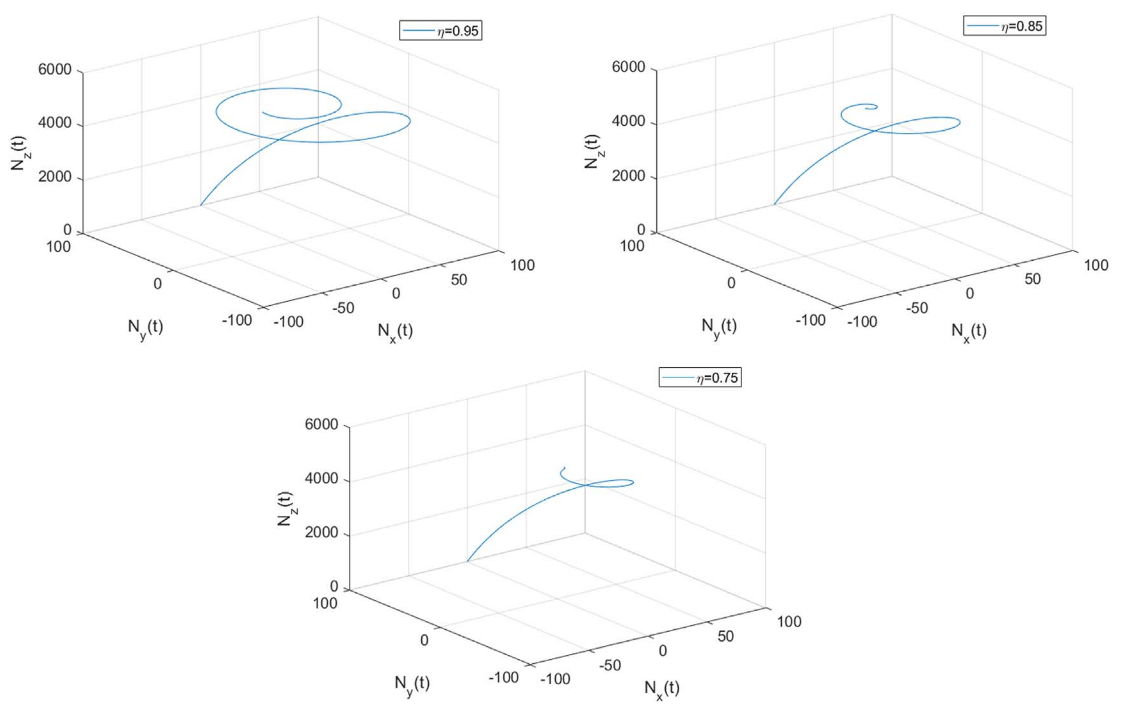

Figure 7. The three-dimensional phase portraits of the solution of system (13), for different values of η and $\Phi(t) = e^t$ with initial conditions $N_x(0) = N_z(0) = 0,\ N_y(0) = 100$ for and $T_1' = 1\ (s)^\eta$, $T_2' = 20\ (ms)^\eta$, $\lambda_0^* = 1$ converging to the balance value N_0. It is evident that reducing the value of η serves as a stabilizing mechanism for the system described by equations (13).

Table 2. Comparison of the exact solution and approximate solution obtained by the present method and the proposed methods in [26], [43], and [25] when $\eta = 1$ and $t \in [0, 1]$.

$N_i(t)$	t	Exact soution	Present method	Kumar *et al* [26]	Petras [43]	Singh [25]
	0.1	9.9335	9.9335	9.9335	9.2237	9.9245
	0.3	29.1120	29.1120	29.1034	29.0937	29.1080
$N_x(t)$	0.5	46.7588	46.7588	46.6823	46.7507	46.7732
	0.7	62.2060	62.2060	61.8762	62.1921	62.2180
	0.9	74.8859	74.8859	73.8911	74.8606	74.8814
	0.1	99.0042	99.0042	99.0187	99.0051	99.0213
	0.3	94.1113	94.1113	94.1837	94.1166	94.1645
$N_y(t)$	0.5	85.5915	85.5915	85.5518	85.5942	85.5689
	0.7	73.8536	73.8536	73.1630	73.8635	73.7886
	0.9	59.4258	59.4258	57.0572	59.4296	59.3782
	0.1	9.5163	9.5163	0.0952	0.0952	0.0952
	0.3	25.9182	25.9182	0.2592	0.2590	0.2592
$N_z(t)$	0.5	39.3469	39.3469	0.3935	0.3934	0.3935
	0.7	50.3415	50.3415	0.5034	0.5033	0.5034
	0.9	59.343	59.343	0.5934	0.5934	0.5934

Table 3. Comparison of the MAE of the present method and the proposed methods in [26], [43], and [25] when $\eta = 1$ and $t \in [0, 1]$.

$N_i(t)$	Present method	Kumar *et al* [26]	Petras [43]	Sing [25]
$N_x(t)$	7.1922×10^{-17}	1.5849	9.5100×10^{-2}	1.6200×10^{-2}
$N_y(t)$	3.2971×10^{-16}	3.7682	4.0500×10^{-2}	7.9300×10^{-2}
$N_z(t)$	6.5218×10^{-23}	9.8900×10^{-5}	7.3754×10^{-4}	5.0663×10^{-6}

Figures 1 and 2 show decaying oscillations of $N_x(t)$ and $N_y(t)$ for $\Phi(t) = t$, $\Phi(t) = t^2$, and $\Phi(t) = e^t$ with different values of $\eta = 0.99,\ 0.7$. It is obvious from the figures that $N_x(t)$ and $N_y(t)$ decay as time increases.

The behavior of $N_z(t)$ with respect to time t is shown in figure 3 for $\Phi(t) = t$, $\Phi(t) = t^2$, and $\Phi(t) = e^t$. It follows that $N_z(t)$ sharply decreases as the parameter η decreases, and it reaches the equilibrium later.

Table 4. The CPU time (seconds) of the present method for $\Phi(t) = t,\ \Phi(t) = t^2$ and $\Phi(t) = e^t$

$N_i(t)$	η	$\Phi(t) = t$	$\Phi(t) = t^2$	$\Phi(t) = e^t$
	0.95	9.426548	13.874438	13.549149
$N_x(t)$	0.85	9.065340	11.751834	12.459779
	0.75	5.748150	6.232823	10.184794
	0.95	10.405756	12.959690	14.441258
$N_y(t)$	0.85	9.523751	12.263305	13.553911
	0.75	7.065278	6.627926	12.597419
	0.95	7.977690	11.545089	16.419121
$N_z(t)$	0.85	7.785860	10.942670	16.558750
	0.75	5.488590	5.914509	14.904016

Figure 4 illustrates the dynamic relationship between the components $N_x(t)$ and $N_y(t)$ for $\Phi(t) = t, \Phi(t) = t^2$, and $\Phi(t) = e^t$, and $\eta = 0.99,\ 0.7$ on the complex plane. An arranged spiral is exhibited for $\eta = 0.99$, and it is concluded that decay happens more rapidly as the value of η decreases.

In figures 5–7, the entire direction is depicted using the three-dimensional phase portraits for $\Phi(t) = t$, $\Phi(t) = t^2$ and $\Phi(t) = e^t$ and $\eta = 0.95,\ 0.85,\ 0.75$ with initial values $N_x(0) = 0,\ N_y(0) = 100$ and $N_z(0) = 0$ converging to the balance value N_0.

Remark 4.1. By selecting different forms of $\Phi(t)$, we offer a generalized memory kernel beyond integer-order models. As Almeida noted in [6], the Φ-Caputo derivative is capable of capturing hidden properties of physical processes by adapting the time scale via Φ. Therefore, the different functions of Φ tasted in this paper, they were selected to demonstrate how the memory structure and relaxation behavior of the Bloch system can be adjusted using various time scales. This concept may reflect different experimental or biological conditions in MRI or NMR.

The case $\Phi(t) = t$ corresponds to the classical Caputo derivative and models memory effects with a uniform time scales. It is consistent with earlier models of fractional Bloch equations proposed by Magin *et al* in [20], where Caputo derivatives were used to capture anomalous relaxation and dispersion in NMR experiments. When $\Phi(t) = t^2$, it implies that time grows in an accelerated manner. This approach is useful for modeling nonlinear magnetic field gradients. This type of time variation has been introduced by Velasco in [39]. This model may be suitable for situations in which the relaxation rate particularly when the parameter T'_2 increases within a magnetic environment. By choosing $\Phi(t) = e^t$, memory effects can exponentially increase or decrease. This timescale is consistent with fast spin processes or relaxation at high energies. The exponential model is capable of simulating rapid decay, some studies on fractional derivatives with exponential kernels (e.g. Caputo-Fabrizio or Atangana-Baleanu derivatives) have shown that the equilibrium of system reaches at a faster rate [45].

Table 2 concludes a comparison between the exact solution of equation (13) and the approximate solution of present method and proposed methods in [26], [43], and [25] for $\Phi(t) = t$, and $\eta = 1$. Also, table 3 shows the maximum absolute error of the present method and the proposed methods in [26], [43], and [25] for $\Phi(t) = t$, and $\eta = 1$.

Table 4 shows CPU times obtained for the solutions of the system of equations (13) for the method presented here in the cases $\Phi(t) = t$, $\Phi(t) = t^2$, and $\Phi(t) = e^t$. The computational time of our method is expected, considering the analytical nature of the solution and the presence of fractional memory in the model.

5. Conclusion

In this paper, we developed a novel procedure for solving the system of Φ-Caputo fractional Bloch equations. The Φ-Caputo fractional derivative provides extra flexibility when analyzing physical samples, thereby revealing some impressive characteristics especially in modeling time-dependent non-uniform memory effects that the ordinary-order derivative cannot portray. Also by changing the function $\Phi(t)$, we can modify the time structure of the model to better match physical realities such as nonlinear memory or anomalous dynamics in magnetic systems. The new procedure combined both the Adomian decomposition and generalized Laplace transform methods, which we called Φ-LADP. The Φ-LADP significantly accelerates the solution process because, the Φ-Laplace transform converts fractional differential equations into simpler algebraic forms, which leads to faster and more efficient solutions. Additionally, the Adomian decomposition method operates on the equations without requiring discretization or linearization techniques, and it does not need a predetermined step size as in Runge-Kutta methods, thereby improving the overall accuracy of the results. This combination reduces the number of iterations required for convergence, optimizing both computational time and memory usage. These features highlight the proposed method's innovation and superiority compared to existing approaches.

Moreover,. We examined these analytical solutions with diverse functions in place of $\Phi(t)$, and values of $0 < \eta \leqslant 1$. Considering $\Phi(t) = t$, the BEs with ηth-order Φ-Caputo fractional derivative reduce to the Bloch equations with ηth-order Caputo fractional derivative, and by choosing the value of η near 1 ($\eta = 0.99$), one obtains the solution of the corresponding ordinary BEs. The solutions of the system of Φ-CFBEs did not attain the balance point with diverse functions in place of $\Phi(t)$ and various values of the fractional order. Therefore, we did not observe an attractor. Also, we demonstrated the usefulness of the proposed method by comparison of the existing methods. This study can be extended in the future by exploring various forms of the function $\Phi(t)$, to achieve more accurate simulation of the dynamic behavior of physical system. This approach has the potential to be applied in the analysis of complex MRI data, especially under varying boundary conditions or changing biological parameters. The proposed Φ-LADP method can be extended to solve fractional equations in two-dimensional or three-dimensional modeling of biomedical engineering, physics, or biological dynamics problems.

Data availability statement

The data cannot be made publicly available upon publication because they are not available in a format that is sufficiently accessible or reusable by other researchers. The data that support the findings of this study are available upon reasonable request from the authors.

Author contributions

M Shahrezaee
Conceptualization (equal), Data curation (equal), Formal analysis (equal), Investigation (equal), Methodology (equal), Software (equal), Visualization (equal), Writing – original draft (equal), Writing – review & editing (equal)

M Arabameri 0000-0003-0269-4547
Software (equal), Supervision (equal), Validation (equal), Visualization (equal), Writing – original draft (equal), Writing – review & editing (equal)

A Ahmadian 0000-0002-0106-7050
Supervision (equal), Validation (equal), Writing – review & editing (equal)

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